

COMPACT CONVOLUTION

BRIAN J. DAY

ABSTRACT. We state a Yoneda-type lemma which leads to various functor categories being compact closed.

Lemma. Given a \mathcal{V} -functor

$$T : \mathcal{A}^{\text{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$$

with \mathcal{A} a \mathcal{V}_f -category, suppose that the canonical map

$$\int^X \int_Y \mathcal{A}(Y, X) \otimes T(X, Y) \longrightarrow \int_Y \int^X \mathcal{A}(Y, X) \otimes T(X, Y)$$

is an isomorphism. Then, for each choice of \mathcal{V} -natural isomorphism

$$\mathcal{A}(Y, X) \cong \mathcal{A}(X, Y)^*,$$

we get an isomorphism

$$\int^X T(X, X) \xrightarrow[\alpha]{\cong} \int_Y T(Y, Y)$$

where α_{XY} is the \mathcal{V} -dinatural composite

$$T(X, X) \xrightarrow{\text{can}} [\mathcal{A}(X, Y), T(X, Y)] \xrightarrow{\cong} \mathcal{A}(Y, X) \otimes T(X, Y) \xrightarrow{\text{can}} T(Y, Y).$$

Example. Let $\mathcal{V} = \mathbf{Vect}_k$ and suppose $\text{ob}(\mathcal{A})$ is finite; then $\int^X S(X, X)$ is absolute for all \mathcal{V} -functors $S : \mathcal{A}^{\text{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$, hence

$$\int^X \int_Y \xrightarrow{\cong} \int_Y \int^X$$

always, so

$$\int^X T(X, X) \xrightarrow{\cong} \int_Y T(Y, Y)$$

for $\mathcal{A} = k_*$ (finite groupoid) or \mathcal{A} a finite dimensional Hopf algebra, etc.

Extension. Suppose $\mathcal{V} = \mathbf{Vect}_k$ and $\mathcal{A} \subset \mathcal{C}$ is \mathcal{V} -dense with $\text{ob}(\mathcal{A})$ finite; then, for suitably continuous \mathcal{V} -functors

$$T : \mathcal{C}^{\text{op}} \otimes \mathcal{C} \longrightarrow \mathcal{V}$$

we have

$$\int^C T(C, C) \xrightarrow{\cong} \int_D T(D, D)$$

for each choice of \mathcal{V} -natural isomorphism $\mathcal{A}(X, Y) \cong \mathcal{A}(Y, X)^*$.

Date: March 27, 2006.

Proof. Both

$$\int^X T(X, X) \xrightarrow{\cong} \int^C T(C, C) \quad \text{and} \quad \int_D T(D, D) \xrightarrow{\cong} \int_Y T(Y, Y)$$

are isomorphisms by hypothesis and Yoneda. \square

Convolution. Let

$$p : \mathcal{A}^{\text{op}} \otimes \mathcal{A}^{\text{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}_f \quad \text{and} \quad j : \mathcal{A} \longrightarrow \mathcal{V}_f$$

be a (commutative) promonoidal category where p , $\mathcal{A}(-, -)$, and j have finite support in each variable separately, and suppose there is a natural isomorphism

$$(*) \quad \int^{XY} j(Y) \otimes p(X, B, Y)^* \otimes p(X, C, A) \cong p(A, B, C)^*.$$

For each \mathcal{V} -functor $G : \mathcal{A} \longrightarrow \mathcal{V}_f$ define $G^* : \mathcal{A} \longrightarrow \mathcal{V}_f$ by

$$G^*(X) = \int^B (GB)^* \otimes \int^Y j(Y) \otimes p(X, B, Y)^*.$$

Then

$$\begin{aligned} (G^* \otimes H)(A) &= \int^{XC} G^* X \otimes HC \otimes p(X, C, A) \quad (\text{by defn of } \otimes) \\ &= \int^{XC} \int^B (GB)^* \otimes \int^Y j(Y) \otimes p(X, B, Y)^* \otimes HC \otimes p(X, C, A) \\ &\cong \int^{BC} (GB)^* \otimes HC \otimes p(A, B, C)^* \quad (\text{by } (*)) \\ &\cong \int_{BC} (GB \otimes p(A, B, C))^* \otimes HC \quad (\text{by the lemma}) \\ &\cong \int_{BC} [GB \otimes p(A, B, C), HC] \\ &= [G, H](A). \end{aligned}$$

So $[\mathcal{A}, \mathcal{V}_f]$ is compact.

Example. If \mathcal{A} has an antipode

$$S : \mathcal{A}^{\text{op}} \longrightarrow \mathcal{A},$$

with $S^2 \cong 1$, then $(*)$ holds if both

$$p(X, Y, Z)^* \cong p(SX, SY, SZ) \quad \text{and} \quad p(X, Y, SZ) \cong p(Y, Z, SX).$$

Proof.

$$\begin{aligned}
& \int^{XY} j(Y) \otimes p(X, B, Y)^* \otimes p(X, C, A) \\
& \cong \int^{XY} j(Y) \otimes p(SX, SB, SY) \otimes p(X, C, A) && \text{by hyp,} \\
& \cong \int^{XY} j(Y) \otimes p(SB, Y, X) \otimes p(X, C, A) && \text{by hyp and } S^2 \cong 1, \\
& \cong \int^X \mathcal{A}(SB, X) \otimes p(X, C, A) && \text{since } j * p \cong \mathcal{A}(-, -), \\
& \cong p(SB, C, A) && \text{by Yoneda,} \\
& \cong p(SA, SB, SC) && \text{by hyp and } S^2 \cong 1, \\
& \cong p(A, B, C)^* && \text{by hyp.} \quad \square
\end{aligned}$$

A special case of the above is where $\text{ob}(\mathcal{A})$ is finite. In the literature (cf. [1] and [2]) the situation occurs where $\mathcal{A} \subset \mathcal{C}$ with $(\mathcal{C}, \otimes, I)$ a commutative compact closed category and \mathcal{A} has the trace promonoidal structure induced by \mathcal{C} and $I \in \mathcal{A}$; that is

$$p(X, Y, Z) = \mathcal{C}(X \otimes Y, Z) \quad \text{and} \quad j(X) = \mathcal{C}(I, X).$$

Of course we suppose that $\mathcal{A}(X, Y) = \mathcal{A}(Y, X)^*$, but we must also suppose that

$$\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(Z, X \otimes Y)^*$$

naturally in $X, Y, Z \in \mathcal{A}$ in order for

$$\begin{aligned}
p(SX, SY, SZ) &= \mathcal{C}(SX \otimes SY, SZ) \\
&= \mathcal{C}(Z, X \otimes Y) \\
&= \mathcal{C}(X \otimes Y, Z)^* \\
&= p(X, Y, Z)^*,
\end{aligned}$$

and

$$\begin{aligned}
p(X, Y, SZ) &= \mathcal{C}(X \otimes Y, SZ) \\
&= \mathcal{C}(X \otimes Y \otimes Z, I) \\
&= \mathcal{C}(Y \otimes Z, SX) \\
&= p(Y, Z, SX).
\end{aligned}$$

The point is that the empirical \mathcal{A} , being finite, tends not to be closed under the \otimes of \mathcal{C} .

Example. Let $\mathcal{V} = \mathbf{Vect}_k$. Suppose $\mathcal{A} \subset \mathcal{C}$ is Cauchy dense; then

$$[\mathcal{C}, \mathcal{V}_f] \simeq [\mathcal{A}, \mathcal{V}_f].$$

If \mathcal{C} is a commutative compact closed category and $\text{ob}(\mathcal{A})$ is finite then $[\mathcal{A}, \mathcal{V}_f]$ is compact closed (as we saw earlier) so $[\mathcal{C}, \mathcal{V}_f]$ is compact.

REFERENCES

- [1] Reinhard Häring-Oldenburg, Reconstruction of weak quasi hopf algebras, J. Alg. 194 (1997), pp. 14–35.
- [2] G. Moore and N. Seiberg, Classical and quantum conformal field theory, Communications Math. Phys. 123 (1989), pp. 177–254.

CENTRE OF AUSTRALIAN CATEGORY THEORY, MACQUARIE UNIVERSITY, NSW, 2109, AUSTRALIA